

Math 3280
Review Problems for Exam I

This is not an exhaustive list of all possible types of problems. See the class notes, the textbook and homework for additional problems.

1. Solve the IVP $x' + 2x = t e^{-2t}$, $x(1) = 0$.
2. Find the explicit solution of the IVP $\frac{dx}{dt} = \frac{3t^2 + 4t + 2}{2(x-1)}$, $x(0) = 2$.
3. Find the implicit solution of $\frac{dx}{dt} = \frac{x \cos t + 2t e^x}{1 - \sin t - t^2 e^x}$.
4. Find the explicit solution $x = x(t)$ of $(t^2 + 3tx + x^2) dt - t^2 dx = 0$. Assume $t > 0$.
5. Consider $x'(t) = x - \sqrt[3]{4x}$. Find its equilibrium solutions. Classify the equilibrium solutions and draw its phase line.
6. Consider $x'(t) = x^2(1 - x^2)$. Find its critical points and classify them using the linearization theorem. Draw its phase line.
7. Consider $x'(t) = x^4 - 2x^3 + x^2$. Find its equilibrium points and classify them. Determine the intervals, (initial) x -values, on which the solution $x(t)$ would be increasing or decreasing. Determine the intervals, x -values, on which the solution $x(t)$ would be concave up or down. Use these information to graph several representative flows (solutions) in the tx -plane.
8. Solve the IVP $6x''(t) = 5x'(t) - x(t)$, $x(0) = 4$, $x'(0) = 0$ by first converting it to a first order planar system.
9. Consider the nonlinear system

$$\begin{aligned}\frac{dx}{dt} &= xy + 1 \\ \frac{dy}{dt} &= x^2 + 2\end{aligned}$$

Find all places in the xy -plane that its solution trajectories have (a) Horizontal tangent lines, (b) Vertical tangent lines, or (c) Tangent lines with slope $m = 2$, in any.

10. Solve

$$\begin{aligned}\frac{dx}{dt} &= -xy + e^{t^2} \\ \frac{dy}{dt} &= -\frac{y}{t}\end{aligned} \quad (x(1), y(1)) = \left(\frac{e}{2}, 1\right)$$

Assume $t > 0$.

(Partial) Answers to the Review Problems for Exam I Continued

17. $\lambda = \lambda_1 = \lambda_2 = -1$, $V_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $V_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$; $X(t) = c e^{-t} V$ for any vector V . All solutions are on lines through origin. The critical point $(0, 0)$ is a sink.
18. $\lambda = i$, $V = \begin{pmatrix} 3 - i \\ 5 \end{pmatrix}$; $X(t) = c_1 \begin{pmatrix} 3 \cos t + \sin t \\ 5 \cos t \end{pmatrix} + c_2 \begin{pmatrix} 3 \sin t - \cos t \\ 5 \sin t \end{pmatrix}$. The critical point $(0, 0)$ is a center. Solution curves have clockwise rotation.
19. $\lambda = 1 + 2i$, $V = \begin{pmatrix} -i \\ 1 \end{pmatrix}$; $X(t) = c_1 \begin{pmatrix} e^t \sin 2t \\ e^t \cos 2t \end{pmatrix} + c_2 \begin{pmatrix} -e^t \cos 2t \\ e^t \sin 2t \end{pmatrix}$. The critical point $(0, 0)$ is a spiral source. Solution curves have clockwise rotation.
20. $\lambda_1 = -2$, $V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$; $\lambda_2 = -1$, $V_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$; $X(t) = c_1 e^{-2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. The lines $y = x$ and $y = 2x$ are the straight-line solutions (stable manifolds); The critical point $(0, 0)$ is a sink.

11. Consider the nonlinear system

$$\begin{aligned}\frac{dx}{dt} &= 3y^2(x^2 + 1) \\ \frac{dy}{dt} &= 2x\end{aligned}$$

Solve it to find y as a function of x by considering $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$.

12. Find the solution of $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ with initial values $x(0) = 2$ and $y(0) = 1$

In the following problems, consider the system $X'(t) = AX$, with the given matrix A , and do the following.

- (a) Find its general solution.
- (b) Find its critical point(s) and classify them, if possible.
- (c) Discuss information useful for drawing its representative trajectories, for example,
 - Straight line (invariant) solutions.
 - Slopes of tangent lines to the solution curves.
 - Direction field or the direction of the increasing t value of the solution curves.
 - Converting the solution to rectangular coordinates.
- (d) Draw several representative trajectories.

Note: For these review problems, you may use Mathematica to find the eigenvalues and the corresponding eigenvectors.

$$13. A = \begin{pmatrix} 1 & -1 \\ -2 & 0 \end{pmatrix} \quad 14. A = \begin{pmatrix} 3 & -1 \\ 2 & 0 \end{pmatrix} \quad 15. A = \begin{pmatrix} 3 & -1 \\ 4 & -1 \end{pmatrix} \quad 16. A = \begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix}$$

$$17. A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad 18. A = \begin{pmatrix} -3 & 2 \\ -5 & 3 \end{pmatrix} \quad 19. A = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \quad 20. A = \begin{pmatrix} -3 & 1 \\ -2 & 0 \end{pmatrix}$$

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(Partial) Answers to the Review Problems for Exam I

1. $x(t) = \frac{1}{2}(t^2 - 1)e^{-2t}$
2. $x(t) = 1 + \sqrt{t^3 + 2t^2 + 2t + 1}$
3. $x - x \sin t - t^2 e^x = c$
4. $x(t) = \frac{(1+c)t + t \ln t}{c + \ln t}$
5. $x = -2$, unstable; $x = 0$, stable; $x = 2$, unstable
6. $x = -1$, unstable; $x = 0$, semistable; $x = 1$, stable
7. $x = 0$ and $x = 1$, both semistable; x is increasing for $x < 0$, $0 < x < 1$, and $x > 1$; x is concave downward for $x < 0$ and $\frac{1}{2} < x < 1$; x is concave upward for $0 < x < \frac{1}{2}$ and $x > 1$
8. $x = 12e^{\frac{1}{3}t} - 8e^{\frac{1}{2}t}$
9. No horizontal tangent lines; Vertical tangent lines on the curve $y = -\frac{1}{x}$; Tangent lines with slope $m = 2$ on the lines $x = 0$ and $y = \frac{x}{2}$
10. $x(t) = \frac{e^{t^2}}{2t}$, $y(t) = \frac{1}{t}$
11. $y = \sqrt[3]{c + \ln(x^2 + 1)}$
12. $x(t) = (t + 2)e^{2t}$, $y(t) = e^{2t}$
13. $\lambda_1 = -1$, $V_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$; $\lambda_2 = 2$, $V_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$; $X(t) = c_1 e^{-t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$. The lines $y = 2x$ and $y = -x$ are the straight-line solutions (stable and unstable manifolds); The critical point $(0, 0)$ is a saddle point.
14. $\lambda_1 = 1$, $V_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$; $\lambda_2 = 2$, $V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$; $X(t) = c_1 e^t \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. The lines $y = 2x$ and $y = x$ are the straight-line solutions (unstable manifolds); The critical point $(0, 0)$ is a source.
15. $\lambda = \lambda_1 = \lambda_2 = 1$, $V = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$; $X(t) = c_1 e^t \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^t \left(t \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right)$; The lines $y = 2x$ is the straight-line solution (unstable manifolds); The critical point $(0, 0)$ is a source; Solutions curves try to spiral about the critical point, but the unstable manifold gets in the way.
16. $\lambda_1 = -1$, $V_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$; $\lambda_2 = 0$, $V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$; $X(t) = c_1 e^{-t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. All points on the line $y = x$ are critical points. All other solutions are lines with slope $m = 2$ that move toward the line $y = x$.